University College London
DEPARTMENT OF MATHEMATICS
Mid-Sessional Examinations 2010
Mathematics 1201
Monday 11 January 2010 3.30 – 5.30

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination*

1) (i) Replace the formula $(q \wedge \neg p) \vee \neg(p \wedge \neg r)$ by an equivalent one which does not involve $\neg$, $\wedge$ or $\vee$.

(ii) Negate the formula $(\forall x)\Big((\exists y)P(x,y) \wedge (\forall y)\neg Q(x,y)\Big)$ and replace it by an equivalent formula which does not involve either $\neg$ or $\forall$.

Let $f : A \to B$ be a mapping between sets A, B. Explain what is meant by saying that (a) f is injective ; (b) f is bijective ; (c) f is invertible.
Prove that a mapping is invertible if and only if it is bijective.

Decide with proof whether the mapping $f : \mathbf{Z} \to \mathbf{Z}$, $f(x) = x^3 + x$ is surjective.

Justifying your statement in each case, decide

(iii) for which, if any, $\alpha \in \mathbf{R}$ the mapping $g : \mathbf{R} \to \mathbf{R}$; $g(x) = x^3 + \alpha x$ is injective ;

(iv) for which, if any, $\alpha \in \mathbf{C}$ the mapping $h : \mathbf{C} \to \mathbf{C}$; $h(x) = x^3 + \alpha x$ is injective.

PLEASE TURN OVER

2) Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a subset of a vector space V. Explain what is meant by saying that $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent.

Decide with proof whether the vectors below are linearly independent. If they are not, give an explicit dependence relation between them.

$$\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

Suppose that the subset $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ spans V and that $\mathbf{v}_n$ can be expressed as a linear combination

$$\mathbf{v}_n = \lambda_1 \mathbf{v}_1 + \ldots + \lambda_{n-1} \mathbf{v}_{n-1}.$$

Show that $\{\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}\}$ also spans V.

For the matrix A below, find A^{-1} and express A^{-1} as a product of elementary matrices; hence also express A as a product of elementary matrices.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 4 & -3 \\ 0 & 2 & -1 \end{pmatrix}.$$

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3) Let V, W be vector spaces over a field $\mathbf{F}$ and let $T : V \to W$ be a linear mapping; explain what is meant by

(a) the kernel, $\mathrm{Ker}(T)$ and (b) the image, $\mathrm{Im}(T)$.

State without proof a relationship which holds between $\dim \mathrm{Ker}(T)$ and $\dim \mathrm{Im}(T)$.

Find the general solution to the system $A\mathbf{x} = \mathbf{b}$ when

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 0 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 & -1 & -1 \\ 1 & 1 & 2 & 1 & -3 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}.$$

Denoting by $T_A : \mathbf{Q}^7 \to \mathbf{Q}^4$ the linear map $T_A(\mathbf{x}) = A\mathbf{x}$ find also

(i) a basis for $\mathrm{Ker}(T_A)$ and (ii) a basis for $\mathrm{Im}(T_A)$.

4) Let $T : U \to V$ be a linear map between vector spaces U, V, and let $\mathcal{E} = (e_i)_{1 \le i \le m}$ be a basis for U and $\Phi = (\varphi_j)_{1 \le j \le n}$ be a basis for V.

Explain what is meant by the matrix $\mathcal{M}(T)^{\Phi}_{\mathcal{E}}$ of T taken with respect to $\mathcal{E}$ (on the left) Φ (on the right).

If $S : V \to W$ is also linear and $\Psi = (\psi_k)_{1 \le k \le p}$ is a basis for W prove that

$$\mathcal{M}(S \circ T)^{\Psi}_{\mathcal{E}} = \mathcal{M}(S)^{\Psi}_{\Phi} \mathcal{M}(T)^{\Phi}_{\mathcal{E}}.$$

Let $T : \mathbf{Q}^3 \to \mathbf{Q}^3$ be the mapping $T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - x_3 \\ 2x_1 + 5x_2 + 2x_3 \\ -2x_1 - 3x_2 \end{pmatrix}$

and let $\mathcal{E} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\Phi = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \right\}$.

Write down (i) $\mathcal{M}(T)^{\mathcal{E}}_{\mathcal{E}}$ and (ii) $\mathcal{M}(\mathrm{Id})^{\mathcal{E}}_{\Phi}$. Hence find $\mathcal{M}(T)^{\Phi}_{\Phi}$.

PLEASE TURN OVER

5) Let V be the vector space consisting of all functions $f : \mathbf{R} \to \mathbf{R}$ of the form

$$f(x) \quad = \quad \lambda_1 \sin(x) + \lambda_2 \cos(x) + \lambda_3\, x\sin(x) + \lambda_4\, x\cos(x) \qquad (\lambda_i \in \mathbf{Q})$$

and let $D : V \to V$ be the linear map

$$D(f) = \frac{df}{dx}.$$

Taking $\{ \sin(x),\ \cos(x),\ x\sin(x),\ x\cos(x) \}$ as basis for V find :

i) the matrix of D ; ii) the matrix of D^3 ; iii) the matrix of D^{-1}.

Hence *without further explicit differentiation or integration* write down

iv) $\dfrac{d^3}{dx^3}(2\sin(x) - 3\cos(x) + x\sin(x))$

v) $\displaystyle\int \{ 2\sin(x) - 3\cos(x) + x\sin(x) \} \, dx$

[You may ignore the constant of integration in v)].

6) Let σ be a permutation of the set $\{1, \ldots, n\}$. Explain what is meant by saying

(i) σ is an adjacent transposition ; (ii) σ is a cycle of length m.

Prove that any permutation can be written as a product of adjacent transpositions.

Define $\text{sign}(\sigma)$, and prove that if σ is a cycle of length m then

$$\text{sign}(\sigma) = (-1)^{m-1}.$$

Decompose the following permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\ 11 & 5 & 7 & 13 & 12 & 10 & 2 & 9 & 1 & 8 & 6 & 3 & 4 \end{pmatrix}$$

into a product of disjoint cycles and hence compute $\text{ord}(\sigma)$ and $\text{sign}(\sigma)$.

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